PRODUCT BLOCK FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND WITH SINGULAR KERNEL

حزم المضروبات لمعادلات " فولترا " التكاملية من النوع الثاني ذات النواة المفردة

by

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ملخص البحث: تعتبر طريقة الحزم من أفضل الطرق ذاتية البدء لحل معادلة فولترا ذات النواة المتصلة. في هذا البحث استخدمت هذه الطريقة لحل معادلات فولترا ذات النواة المفردة مع اقتراح تعديلين للطريقة. فقد استخدمت فكرة المضروبات مع الحزم وهذا هو التطوير الأول أما الثاني فهو استخدام التقسيم المتدرج. وقد وضع خوارزم للحل وأظهر تحسن الدقة مع التقسيم المتدرج مقارنة بالتقسيم المتساوي.

Abstract: The Block method is a good self-starting method for solving Volterra Integral Equations with continuous kernel. In this paper two modifications are suggested to develop Block method and to use it to solve Volterra Integral Equations of the second kind with singular kernel. The idea of the product technique is used for Block method, which is the first modification, while the second modification is the use of the graded nodes with Block method. Some algorithms are considered and the solution procedures are carried out. Implementation and testing of the considered algorithms have been done. The results show that the graded nodes give good results compared with equal spaced nodes.

Keywords : VOLTERRA INTEGRAL EQUATION, PRODUCT INTEGRATION, SINGULAR KERNEL, BLOCK METHOD.

1. Introduction: Mathematical modeling processes in the biological and the physical applications (see Brunner [4]) lead quite frequently to Volterra integral Equation of the second kind of the following form:

$$y(x) = f(x) + \int_0^x p(x,t)\bar{k}(x,t,y(t))dt, \qquad x\in[0,T]$$

where $p(x, t)$ refers to the type of singularity used with initial condition of $y(0) = f(0)$ and $\bar{k}$ is smooth on the domain $0\le t \le x \le T$. In this, paper two types of singularities are considered:

1- $P(x,t) = (x-t)^{-\alpha}$ which has a singularity at $x = t$, $0<\alpha <1$

2- $P(x,t) = t^{\alpha-1} (x-t)^{-\alpha}$, $0<\alpha <1$ which has two types of singularities , the first at $t = 0$ and the other is moving along the line $x = t$. The commonly encountered values of α are (**0 and 1/2**), but other (rational) values (e.g., α = **2/3**) are also known to occur, for example in the modeling of the flow of a hot gas through a metallic tube, with reaction arising at the walls of the tube; (see Delves and Walsh [6], Atkinson [1] and Claus [5]). In this paper the product Block is considered.

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2. *Mathematical Preliminaries:* The singular Volterra integral equation of the second kind:

$$y(x) = f(x) + \int_0^x p(x,t)\bar{k}(x,t,y(t))dt, \qquad x\in[0,T] \qquad (1)$$

is assumed to satisfy the conditions for a unique solution (see, for example, Tricomi [9] or Smithies [7]). The interval [0,T] is divided into N = 2M subintervals. The nodes as reported by Attia [3], [4] are chosen to satisfy the following:

$$0 = t_0 < t_1 < t_2 \dots\dots < t_{N-1} < t_N = T$$

$$t_{2k} = \left[\frac{2kT}{N}\right]^{\beta} = \left[\frac{kT}{M}\right]^{\beta}, \quad k = 0,1,2,\dots,M$$

$$t_{2k+1} = \tfrac{1}{2}\left[t_{2k} + t_{2k+2}\right], \quad k = 0,1,\dots,M-1$$

$$h_k = t_{k+1} - t_k \qquad k = 0,1,\dots,2M-1$$

3. *Product Integration*: The product-integration method is based on approximating the smooth function $\bar{k}$ by a polynomial. More precisely, the integral is approximated by:

$$\int_0^x p(x_i,t)\,\bar{k}(x_i,t,y(t))dt \cong \sum_{j=1}^{i} w_{ij}\bar{k}(x_i,t_j,y_j) \qquad (2)$$

where $x_i = t_i$, $i = 1, \dots\dots, N$ and $p(x_i, t) = (x_i - t)^{-\alpha}$ which is the singular part. The weights depend on the quadrature points x_i. Here, a product integration forms Block a method, approximates the integral term.

3.1. Product Block: In this method equation (1) can be written in the following form:

$$y_{2i+1} = f_{2i+1} + \int_0^{x_{2i}} p(x_{2i+1},t)\bar{k}(x_{2i+1},t,y(t))dt + \int_{x_{2i}}^{x_{2i+1}} p(x_{2i+1},t)\bar{k}(x_{2i+1},t,y(t))dt \qquad (3.1)$$

$$y_{2i+2} = f_{2i+2} + \int_0^{x_{2i}} p(x_{2i+2},t)\bar{k}(x_{2i+2},t,y(t))dt + \int_{x_{2i}}^{x_{2i+2}} p(x_{2i+2},t)\bar{k}(x_{2i+2},t,y(t))dt \qquad (3.2)$$

Then, the above two equations can be written in the following approximated form:

$$y_{2i+1} = f_{2i+1} + \sum_{j=0}^{2i} w_{ij}(x_{2i+1} - t_j)^{-\alpha}k(x_{2i+1},t_j,y(t_j)) + w_{2i+1,2i}k(x_{2i+1},t_{2i})y_{2i} + w_{2i+1,2i+1}k(x_{2i+1},t_{2i+1})y_{2i+1} + w_{2i+1,2i+2}k(x_{2i+1},t_{2i+2})y_{2i+2} \qquad (4)$$

and

$$y_{2i+2} = f_{2i+2} + \sum_{j=0}^{2i} \overline{w}_{ij}(x_{2i+2} - t_j)^{-\alpha}k(x_{2i+2},t_j,y(t_j)) + w_{2i+2,2i}k(x_{2i+2},t_{2i})y_{2i} + w_{2i+2,2i+1}k(x_{2i+2},t_{2i+1})y_{2i+1} + w_{2i+2,2i+2}k(x_{2i+2},t_{2i+2})y_{2i+2} \qquad (5)$$

The weights can be calculated by the following technique:

3.2. Calculation of weights: In general if g(x) is singular in [a, b] and f(x) is a smooth function in the same interval then,

$$\int_a^b g(x)f(x)dx = p_1 f(x_1) + p_2 f(x_2) + p_3 f(x_3) \quad x_1, x_2 \text{ and } x_3 \in [a,b]$$

where p_1, p_2 and p_3 can be calculated exactly from the solution of the system of equations:

$$\int_a^b g(x)z(x)dx = p_1 z(x_1) + p_2 z(x_2) + p_3 z(x_3),$$

$$z(x) = x^k \text{ and } k = 0,\ 1,\ 2.$$

This method is called Product-3 [2]. Then the weights are:

$$\int_{x_{2i}}^{x_{2i+1}} (x_{2i+1} - t)^{-\alpha} f(t)dt = w_{2i+1,2i} f_{2i} + w_{2i+1,2i+1} f_{2i+1} + w_{2i+1,2i+2} f_{2i+2}$$

$$\int_{x_{2i}}^{x_{2i+2}} (x_{2i+2} - t)^{-\alpha} f(t)dt = w_{2i+2,2i} f_{2i} + w_{2i+2,2i+1} f_{2i+1} + w_{2i+2,2i+2} f_{2i+2},$$

where:

$$w_{v,2i+2} = \frac{abR_1 - R_2(a+b) + R_3}{(c-a)(c-b)}, \quad w_{v,2i+1} = \frac{a-c}{b-a} w_{v,2i+2} + \frac{R_2 - aR_1}{b-a},$$

$$w_{v,2i} = -w_{v,2i+1} - w_{v,2i+2} + R_1, \quad R_1 = \frac{(x_v - x_{2i})^{1-\alpha}}{1-\alpha},$$

$$R_2 = (x_v - x_{2i})^{1-\alpha}\left[\frac{x_v}{1-\alpha} - \frac{(x_v - x_{2i})}{2-\alpha}\right] \text{ and}$$

$$R_3 = (x_v - x_{2i})^{1-\alpha}\left[\frac{(x_v - x_{2i})^2}{3-\alpha} - \frac{2x_v(x_v - x_{2i})}{2-\alpha} + \frac{x_v^2}{1-\alpha}\right]$$

where $a = t_{2i}$, $b = t_{2i+1}$, $c = t_{2i+2}$ and $v = 2i+1$, $2i+2$ for the first and the second integral respectively . The weights are w_{ij} , $\overline{w}_{ij}$ can be calculate as follows:

$$\int_{x_{2j-2}}^{x_{2j}} (x_{2i+1} - t)^{-\alpha} f(t)dt = w_{2i+1,2j-2} f_{2j-2} + w_{2i+1,2j-1} f_{2j-1} + w_{2i+1,2j} f_{2j} \quad \text{and}$$

$$\int_{x_{2j-2}}^{x_{2j}} (x_{2i+2} - t)^{-\alpha} f(t)dt = \overline{w}_{2i+2,2j-2} f_{2j-2} + \overline{w}_{2i+2,2j-1} f_{2j-1} + \overline{w}_{2i+2,2j} f_{2j}$$

then,

$$w_{v,2j} = \frac{abf_1 - f_2(a+b) + f_3}{(c-a)(c-b)}, \quad w_{v,2j-1} = \frac{a-c}{b-a} w_{v,2j} + \frac{f_2 - af_1}{b-a} \quad \text{and}$$

$$w_{v,2j-2} = -w_{v,2j} - w_{v,2j-1} + f_1$$

where

$j = 1, 2, 3, \ldots\ldots\ldots\ldots, i$, $a = t_{2j-2}$, $b = t_{2j-1}$, $c = t_{2j}$ and

$$f_1 = \frac{(x_v - x_{2j-2})^{1-\alpha}}{1-\alpha} - \frac{(x_v - x_{2j})^{1-\alpha}}{1-\alpha},$$

$$f_2 = (x_v - x_{2j})^{1-\alpha}\left[\frac{(x_v - x_{2j})}{2-\alpha} - \frac{x_v}{1-\alpha}\right] - (x_v - x_{2j-2})^{1-\alpha}\left[\frac{(x_v - x_{2j-2})}{2-\alpha} - \frac{x_v}{1-\alpha}\right]$$

$$f_3 = (x_v - x_{2j})^{1-\alpha}\left[-\frac{(x_v - x_{2j})^2}{3-\alpha} + \frac{2x_v(x_v - x_{2j})}{2-\alpha} - \frac{x_v^2}{1-\alpha}\right]$$

$$-(x_v - x_{2j-2})^{1-\alpha}\left[-\frac{(x_v - x_{2j-2})^2}{3-\alpha} + \frac{2x_v(x_v - x_{2j-2})}{2-\alpha} - \frac{x_v^2}{1-\alpha}\right]$$

where v=2i+1 for w_{ij} and v=2i+2 for $\bar{w}_{ij}$

Equations (4) and (5) are solved to get y_{2i+1}, y_{2i+2}.

3.3 Product Block Algorithm: This algorithm deals with linear Volterra integral equations with singular kernel.

Algorithm	To find a solution to linear Volterra integral equations with singular kernel of type $k(x,t)=(x-t)^{-\alpha}\ \bar{k}(x,t)$ with variable and constant step length by Product Block.
INPUT	Number of intervals (N), Alpha (α) and Beta (β).
OUTPUT	Approximate Solutions y_i, root mean square error ($E_{r.m.s}$), maximum error (E_{max}) and position of maximum error ($X_{e.m}$)
Step 1	**y[0]=f(0), M=N/2; x[0]=0, t[0]=0.**
Step 2	**For i=1 to M-1 do steps (3-4).**
Step 3	**k=2i, x[k]=(k/N)$^{\beta}$.**
Step 4	**T[k]=x[k], j=2i-1** **H[i]=(x[j+1] - x[j-1])/2** **x[j]=x[j-1]+H[i]** **T[j]=x[j].**
Step 5	**x[n]=1, T[n]=1, H[M]=(x[n]-x[n-2])/2** **x[n-1]=x[n-2]+H[M], T[n-1]=x[n-1]**
Step 6	For **i=0** to **M-1** do steps (7-16).
Step 7	Calculate **w[2i+1,2i], w[2i+1,2i+1], w[2i+1,2i+2].** **w[2i+2,2i], w[2i+2,2i+1], w[2i+2,2i+2]** using Product-3.
Step 8	Calculate $b_1, b_2, b_3, b_4, b_5, b_6$.
Step 9	$L_1 = b_2\ b_6 - b_3\ b_5$, $L_2 = b_3\ b_4 - b_1\ b_6$, $b = b_2\ b_4 - b_1\ b_5$ $y[2i+1] = L_1/b$ $y[2i+2] = L_2/b$
Step 10	If **(i<M-1)** then do steps (11-16).
Step 11	**b_3=0, b_6=0.**
Step 12	For **L=i+1,** down to **1** do steps (13-16).
Step 1	Calculate **w[2i+1,2l], w[2i+1,2l-1], w[2i+1,2l-2]** **w[2i+2,2l], w[2i+2,2l-1], w[2i+2,2l-2]** using Product-3.
Step 14	For **j=2l - 2,** to **2l** do steps (15-16).
Step 15	**b_3=b_3+w[2i+1,j] k(x[2i+1], t[j]) y[j].**
Step 16	**b_6=b_6+w[2i+2,j] k(x[2i+2], t[j]) y[j].**
Step 17	OUTPUT. Stop.

For non-linear Volterra integral equations with singular kernel, Newton-Raphson method is used to solve the non-linear equation and step 9 is changed as follow:

Step9 **p_0=y[2i], q_0=y[2i]** and call Newton procedure to calculate **y[2i+1], y[2i+2].**

4. Numerical Results: The numerical algorithm described above is used for the computation of the Product Block and tested for several examples.

4.1 Test Examples: The following table shows different examples used in Linear case (These examples were taken from [2], [6], [7]).

Table (1) Examples and its Solutions which are used in Linear case

Ex	Equation	Exact Solution
1	$y(x) = x - \frac{4}{3}x^{3/2} + \int_0^x (x-t)^{-1/2} y(t)dt$	$y(x) = x$
2	$y(x) = f_1(x) - \frac{1}{4}\int_0^x (x-t)^{-1/2} y(t)dt$	$y(x) = \frac{1}{\sqrt{1+x}}$
3	$y(x) = f_2(x) + \int_0^x (x-t)^{-\alpha}(x^2+t^2)y(t)dt$	$y(x) = x$
4	$y(x) = x - \left[\frac{1}{(1-\alpha)} - \frac{1}{(2-\alpha)}\right]x^{2-\alpha} + \int_0^x (x-t)^{-\alpha} t y(t)dt$	$y(x) = x$

where $f_1(x) = \frac{1}{\sqrt{1+x}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-x}{1+x}\right)$ and

$$f_2(x) = x - \left[\frac{1}{(1-\alpha)(2-\alpha)} + \frac{6}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}\right]x^{4-\alpha}$$

4.2 Model Integral Equation with Two Types of Singularities:

$$y(x) = 1 + 5\left(1 - \frac{\Gamma(\alpha)\Gamma(3-\alpha)}{2}\right) + \int_0^x (x-t)^{-\alpha} t^{\alpha-1} y(t)dt,$$

$$y(x) = 2x^5, \qquad 0 < \alpha < 1, \qquad 0 \le x \le 1$$

4.3 Test Example of Non-Linear Case:

$$y(x) = x - \frac{6x^{4-\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)} + \int_0^x (x-t)^{-\alpha} y^3(t)dt,$$

$$y_e = x$$

Results of the root mean square errors, maximum errors and its positions for the different considered examples uses Block method for linear and non-linear cases are shown in Table 2 and Table3, respectively. $p(x_i, t) = (x_i - t)^{-\alpha}$, while the same results using Product Block method for linear case is shown in Table 4 when $p(x_1, t) = t^{\alpha-1}(x-t)^{-\alpha}$.

***Table (2)** Results of the root mean square errors, maximum errors and its position, When $p(x_1, t) = (x - t)^{-\alpha}$ for different examples using Product Block method in Linear case when $p(x,t) = (x-t)^{-\alpha}$*

		Ex.	N α	Graded-Mesh			Uniform Mesh			
				βE_{rms}	X_{EM}	E_M	E_M	X_{EM}	E_{rms}	E_R
1	4	0.5	1.35	1.16×10^{-12}	1.000	1.82×10^{-12}	2.82×10^{-11}	1.00	1.39×10^{-11}	0.08
2	32	0.5	1.30	3.47×10^{-8}	0.249	4.62×10^{-8}	1.36×10^{-7}	0.03	4.70×10^{-8}	0.74
3	32	0.1	1.00	3.35×10^{-7}	0.97	6.98×10^{-7}	6.98×10^{-7}	0.97	3.35×10^{-7}	1.00
	32	0.2	1.00	5.04×10^{-7}	0.97	1.11×10^{-6}	1.11×10^{-6}	0.97	5.04×10^{-7}	1.00
	32	0.3	0.95	8.46×10^{-7}	0.97	2.05×10^{-6}	2.05×10^{-6}	0.97	8.46×10^{-7}	0.99
4	4	0.1	0.90	0.000000	0.000	0.000000	2.73×10^{-11}	0.75	1.36×10^{-12}	0.00
		0.2	0.75	8.39×10^{-13}	0.797	1.82×10^{-12}	1.82×10^{-12}	1.00	9.96×10^{-12}	0.84
		0.3	0.75	4.07×10^{-13}	0.595	9.09×10^{-13}	9.09×10^{-13}	1.00	4.07×10^{-13}	1.00
		0.4	1.45	4.67×10^{-13}	0.513	9.09×10^{-13}	1.82×10^{-12}	0.63	1.27×10^{-12}	0.37
		0.5	1.35	5.35×10^{-13}	0.392	9.09×10^{-13}	1.82×10^{-12}	0.44	1.08×10^{-12}	0.50
		0.6	1.45	4.35×10^{-13}	0.366	9.09×10^{-13}	1.82×10^{-12}	0.63	1.02×10^{-12}	0.43
		0.7	1.25	3.18×10^{-13}	0.846	9.09×10^{-13}	1.82×10^{-12}	0.69	9.25×10^{-12}	0.34
		0.8	1.35	4.98×10^{-13}	0.360	9.09×10^{-13}	1.82×10^{-12}	0.53	9.29×10^{-12}	0.54
		0.9	1.20	4.56×10^{-13}	0.708	9.09×10^{-13}	1.82×10^{-12}	0.63	9.28×10^{-12}	0.50

where E_{rms}: Root Mean Square Error, E_M: Maximum Error,

X_{EM} : Position of maximum error and $E_R = \dfrac{E_{rms} \text{ of Graded Mesh}}{E_{rms} \text{ of Uniform Mesh}}$

***Table (3)** Results of the root mean square errors, maximum errors and its position, for different cases of α using Product Block method in Non-Linear case when $p(x,t) = (x-t)^{-\alpha}$*

		Graded Mesh				Uniform Mesh			
α	N		$E_{rms}\beta$	X_{EM}	E_M	E_M	X_{EM}	E_{rms}	E_R
0.1	8	0.95	7.26×10^{-5}	0.880	1.19×10^{-4}	1.39×10^{-4}	0.875	7.60×10^{-5}	0.956
	16	0.95	4.70×10^{-6}	0.940	8.72×10^{-6}	1.02×10^{-5}	0.938	4.86×10^{-6}	0.968
	32	0.95	3.06×10^{-7}	0.036	6.25×10^{-7}	7.14×10^{-7}	0.969	3.13×10^{-7}	0.978
	64	0.95	1.99×10^{-8}	0.018	4.79×10^{-8}	4.80×10^{-8}	0.984	2.02×10^{-8}	0.984
0.2	8	0.95	1.06×10^{-4}	0.880	1.77×10^{-4}	2.08×10^{-4}	0.875	1.14×10^{-4}	0.925
	16	0.95	7.02×10^{-6}	0.940	1.40×10^{-5}	1.67×10^{-5}	0.938	7.44×10^{-6}	0.944
	32	0.95	4.76×10^{-7}	0.970	1.03×10^{-6}	1.24×10^{-6}	0.969	4.96×10^{-7}	0.960
	64	0.95	3.26×10^{-8}	0.017	7.74×10^{-8}	8.78×10^{-8}	0.984	3.36×10^{-8}	0.972
0.3	8	0.90	1.72×10^{-4}	1.000	3.09×10^{-4}	4.85×10^{-4}	1.000	2.22×10^{-4}	0.773
	16	0.90	1.19×10^{-5}	0.077	2.41×10^{-5}	3.33×10^{-5}	0.938	1.43×10^{-5}	0.835
	32	0.95	8.84×10^{-7}	0.970	2.28×10^{-6}	2.77×10^{-6}	0.969	9.61×10^{-7}	0.919
	64	0.95	6.31×10^{-8}	0.985	1.76×10^{-7}	2.10×10^{-7}	0.984	6.73×10^{-8}	0.938

In the following figure the relation between (- log (E_{rms})) and β is plotted at N = 32. when $p(x,t) = t^{\alpha-1} (x - t)^{-\alpha}$ and λ = 0.2 , - 5, 1.

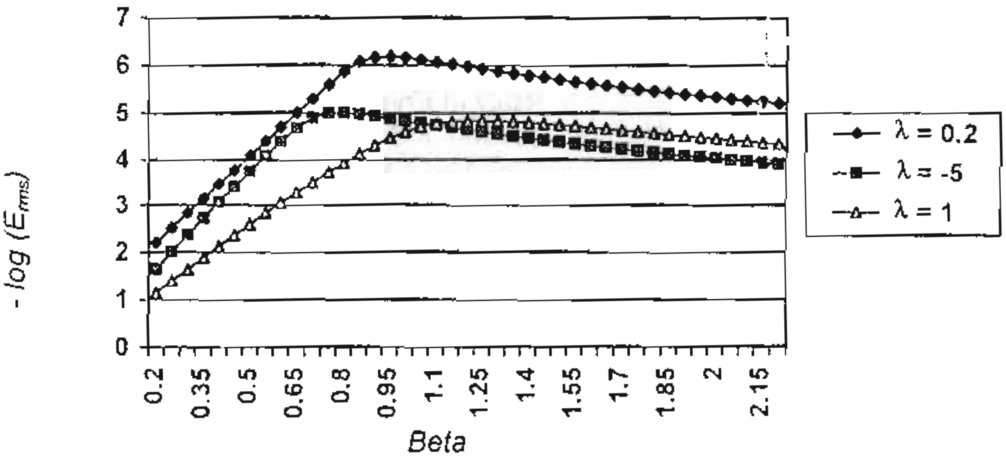

Fig. 1 Relation between (-log E_{rms}) and β at N=32

* The maximum values of - log (E_{rms}) means minimum value of E_{rms}
* When λ = 0.2 the minimum E_{rms} occurs at β = 0.90
* When λ = -5 the minimum E_{rms} occurs at β = 0.80
* When λ = 1 the minimum E_{rms} occurs at β = 1.15

5. Conclusions: The product methods give good results when using it on graded mesh compared with equal spaced nodes at the same number of subintervals. We use the product Block to solve singular Volterra integral equation of the second kind. We conclude to use product Block on graded mesh. The equal spaced nodes can be considered as a special case from the graded mesh. Attia and Nour [3] proposed that the value of β is controlled by two factors: the first factor is the suprimum of the function and the second is the large variation of the function. For the second case, the kernel of this type has two types of singularities, the first at t = 0 and the other is moving along the line x = t. Results of experiment showed that:

1-The product method is a good method to solve this type of singular equations

2 - The optimum value of β in the case of graded meshes is β ∈ (0.7 , 1.5)

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